A note on the convergence of renewal and regenerative processes to a Brownian bridge

Serguei Foss   Takis Konstantopoulos

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Abstract

The standard functional central limit theorem for a renewal process with finite mean and variance, results in a Brownian motion limit. This note shows how to obtain a Brownian bridge process by a direct procedure that does not involve conditioning. Several examples are also considered.

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1 The basic theorem

In proving convergence results for a stochastic ordered graph on the integers [2], we noticed that one can obtain a Donsker-like theorem for Brownian bridge in a somewhat non-standard manner. The result appears to be new. As it may be of potential interest in some related areas (statistics, large deviations), we summarise it in this short note.

Consider a (possibly delayed) renewal process on $[0, \infty)$ with renewal epochs

$$0 < R_1 < R_2 < \cdots .$$

We assume that $\{R_{n+1} - R_n\}_{n \geq 1}$ are i.i.d. with mean μ and variance σ^2, both finite. Let

$$A_t := \#\{n \geq 1 : \ R_n \leq t\}$$

be the associated counting process. The standard functional central limit theorem for a renewal process, see, e.g., [1], states that the sequence of processes $\xi_1, \xi_2, \ldots$, where

$$\xi_n(t) := \frac{A_{nt} - \mu^{-1} nt}{\sqrt{n}}, \quad t \geq 0,$$

converges weakly, as $n \to \infty$, to $\mu^{-3/2}\sigma W$, where W is a standard Brownian motion on $[0, \infty)$. Weak convergence (denoted by $\Rightarrow$ below) means weak convergence of probability

measures on the space $D[0,\infty)$ of functions which are right continuous with left limits, equipped with the usual Skorokhod topology (see, e.g., [3], [7]).

A standard Brownian bridge [3, p. 84] W^0 is defined, in distribution, as a standard Brownian motion W on $[0,1]$, conditional on $W_1 = 0$, i.e. as the weak limit of the sequence of probability measures

$$P(W \in \cdot \mid 0 \le W_1 \le 1/n), \quad n \in \mathbb{N},$$

as $n \to \infty$. Often, when Brownian bridge is obtained as a limit by a functional central limit theorem, there is an *explicit* underlying conditioning that takes place. One first proves convergence to a Brownian motion and uses conditioning to prove convergence to a Brownian bridge. Brownian bridges appear in limits of urn processes, and also in limits of empirical distributions [3, Thm. 13.1].

In this note we remark that it is possible to obtain a Brownian bridge from a renewal process, without the use of conditioning.

Theorem 1. *Define, for $u > 0$,*

$$\eta_u(t) := \frac{R_{[tA_u]} - tu}{\sqrt{u}}, \quad 0 \le t \le 1.$$

Considering η_u as a random element of $D[0,1]$ (equipped with the topology of uniform convergence on compacta), we have

$$\eta_u \Rightarrow \mu^{-1/2}\sigma W^0, \qquad \text{as } u \to \infty,$$

where W^0 is a standard Brownian bridge.

Here, $[x]$ denotes the largest integer not exceeding the real number x. We remark that R_{A_u} is "close" to u, in the sense that $R_{A_u} \le u < R_{1+A_u}$. In fact, the difference $u - R_{A_u}$ (known as the age of the renewal process) is a tight family (over $u \ge 0$) of random variables. In the above theorem, we just introduce another parameter, t, and measure the difference between tu and $R_{[tA_u]}$. When $t = 0$ or 1, this difference is "negligible" with respect to any power of u. When t is between 0 and 1, then the difference is of the "order of $\sqrt{u}$" in the sense that when divided by $\sqrt{u}$ it converges to a normal random variable. Jointly, over all $t \in [0,1]$, we have convergence to a Brownian bridge, and this is what we show next.

Proof. Consider, for $u > 0$,

$$y_u(t) := \frac{R_{[tu]} - \mu tu}{\sqrt{u}}, \quad t \ge 0.$$

From Donsker's theorem [3] for the random walk $\{R_n\}$ we have that $y_u \Rightarrow \sigma W$, where W is a standard Brownian motion. Define also, for $u > 0$,

$$\varphi_u(t) := \frac{tA_u}{u}.$$

From the law of large numbers for the renewal process, $A_u/u \to \mu^{-1}$, a.s., as $u \to \infty$. Hence, φ_u converges a.s. (and weakly) to the deterministic process $\{\mu^{-1}t\}$. Since composition is a continuous function [3] we have that

$$\{(y_u \circ \varphi_u)(t)\} \Rightarrow \{\sigma W_{\mu^{-1}t}\} \stackrel{d}{=} \{\mu^{-1/2}\sigma W_t\}. \tag{1}$$

We also have

$$(y_u \circ \varphi_u)(t) = \frac{R_{[tA_u]} - \mu t A_u}{\sqrt{u}},$$

and so

$$\begin{aligned}\eta_u(t) &= (y_u \circ \varphi_u)(t) + \mu t \frac{A_u - \mu^{-1}u}{\sqrt{u}} \\ &= (y_u \circ \varphi_u)(t) - t(y_u \circ \varphi_u)(1) - t\frac{u - R_{A_u}}{\sqrt{u}}. \end{aligned} \tag{2}$$

Observe now that $\{u - R_{A_u}, u \geq 0\}$ is a tight family. Indeed, from standard renewal theory (see, e.g., [1]), if R_1 has a non-lattice distribution, then $u - R_{A_u}$ converges weakly as $u \to \infty$. And if R_1 has a lattice distribution with span h, then a similar convergence takes places for $nh - R_{A_{nh}}$ as $n \to \infty$. Since, for all $u \geq 0$, $0 \leq u - R_{A_u} \leq ([u/h]+1)h - R_{A_{[u/h]}}$, the family $\{u - R_{A_u}, u \geq 0\}$ is tight even in the lattice case. Tightness implies that the last term of (2) converges to 0 in probability. From the convergence stated in (1) and the decomposition (2), we have that

$$\{\eta_u(t)\}_{0\leq t\leq 1} \Rightarrow \mu^{-1/2}\sigma\{W_t - tW_1\}_{0\leq t \leq 1}.$$

It is well known [4] that a standard Brownian bridge W^0 can be represented as $W_t^0 = W_t - tW_1$, and so the process above is the limit we were looking for. □

2 Extensions, discussion, and examples

Here is a different version that, perhaps, makes Theorem 1 clearer: Suppose that M is a regenerative random measure on $[0, \infty)$. That is, there is some renewal process with points $T_0 < T_1 < T_2 < \cdots$ such that the random measures obtained by restricting M onto $[T_n, T_{n+1})$, $n = 0, 1, 2, \ldots$, are i.i.d. Suppose that

$$\begin{gathered}\mu := E(T_2 - T_1), \quad \operatorname{var}(T_2 - T_1) < \infty, \\ \alpha := EM([T_1, T_2)), \quad 0 < \operatorname{var} M([T_1, T_2)) < \infty.\end{gathered}$$

Define the random distribution function of M by

$$S(t) = M((0, t]), \quad u \geq 0.$$

By the law of large numbers, $S(t)/t \to \mu^{-1}\alpha$, a.s. as $t \to \infty$. Consider the generalised inverse

$$S^{-1}(u) := \inf\{t \geq 0 : \ S(t) > u\}, \quad u \geq 0.$$

Then, in some naive sense, S^{-1} composed with S is "approximately" the identity function, but what can we say about the composition of S^{-1} with a fraction tS of S where $0 < t < 1$? The law of large numbers tells us that, almost surely,

$$\frac{S(tS^{-1}(u))}{u} \xrightarrow[u\to\infty]{} t.$$

An extension of the previous theorem quantifies the deviation:

Theorem 2. *As $u \to \infty$, the sequence of processes η_u where*

$$\eta_u(t) := \frac{S(tS^{-1}(u)) - tu}{\sqrt{u}}, \quad 0 \le t \le 1,$$

converges weakly to a Brownian bridge.

The proof of this is analogous to the previous one, so it is omitted. Observe that the "tying down" of the Brownian motion occurs naturally at $t = 0$ and $t = 1$.

The Brownian bridge has a scaling constant depending on the parameters of the process S.

Note that the regenerative assumption is not crucial. All we need is to have a process for which a Donsker theorem with a Brownian limit holds. This is then translatable to a Brownian bridge limit.

If we interchange the roles of S and S^{-1} we still get a Brownian bridge but with different constant. For instance, interchanging the roles of $\{R_n\}$ and $\{A_u\}$ in Theorem 1 we obtain that

$$\eta'_n(t) := \frac{A(tR_n) - tn}{\sqrt{n}}, \quad 0 \le t \le 1,$$

converges weakly, as $n \to \infty$, to κW^0, where W^0 is a standard Brownian bridge and $\kappa = \sigma\mu^{-1}$.

2.1 An interpretation

To better understand the phenomenon, we cast the limit theorem as follows: We have a random function S, composed with scaling functions

$$\rho_t : x \mapsto tx$$

and composed again with the inverse function S^{-1} and we look at the asymptotic behaviour of the family of random functions

$$S \circ \rho_t \circ S^{-1} - \rho_t, \quad 0 \le t \le 1, \tag{3}$$

(or of $S^{-1} \circ \rho_t \circ S$), as a function of the parameter t. Thus, the time parameter of the Brownian bridge obtained in the limit plays the role of a scaling factor. When t is 0 or 1, $S \circ \rho_t \circ S^{-1} - \rho_t$ is approximately zero (with respect to the normalising factor). This raises the following three questions:
(i) How much "one-dimensional" is this phenomenon?
(ii) Can we replace the family ρ_t by a more general homotopy?
(iii) Are different kind of bridges possible to obtain?
With respect to the latter question, we could start with a regenerative process with finite mean but infinite variance, one that belongs to the domain of attraction of, say, a self-similar Lévy process.

2.2 Four examples

EXAMPLE 1 The first is a simple example involving a standard Brownian motion W. Let X denote the (strong) Markov process

$$X_t = (W_t - t) - \min_{0\le s\le t}(W_s - s), \quad t \ge 0, \tag{4}$$

which is the reflection of the drifted Brownian motion $\{W_t - t\}$. This process in natural in many areas of applied probability, e.g. in the diffusion approximation of a queue. We have $X_0 = 0$, $X_t \ge 0$. The *Brownian area process*

$$S(t) = \int_0^t X_r dr \tag{5}$$

is non-decreasing. Fix some $u \ge 0$ and $t \in [0,1]$. By continuity, there is a unique point between 0 and u that splits the area $S(u)$ into two parts with ratio $t : (1-t)$. Call this point $H_u(t)$. Specifically,

$$H_u(t) := \min\left\{v \ge 0 :\ t\int_0^v X_r dr = (1-t)\int_v^u X_r dr\right\}, \quad 0 \le t \le 1.$$

We then claim that

$$\eta_u(t) := \frac{H_u(t) - tu}{\sqrt{u}}, \quad 0 \le t \le 1,$$

converges weakly to a Brownian bridge as $u \to \infty$. To see this, observe that

$$S^{-1}(x) = \min\{v \ge 0 :\ S(v) = x\},$$

and hence

$$\begin{aligned} S^{-1}(tS(u)) &= \min\{v \ge 0 :\ S(v) = tS(u) \\ &= \min\{v \ge 0 :\ S(v) = t(S(v) + S(u) - S(v))\} \\ &= \min\{v \ge 0 :\ (1-t)S(v) = t(S(u) - S(v))\} = H_u(1-t). \end{aligned}$$

Apply Theorem 2 to get the result. (Notice that $\eta_u(1-t)$ also converges to a Brownian bridge.)

EXAMPLE 2 Same as Example 1, but with W being a *zero-mean* Lévy process. The Brownian bridge in Example 1 was obtained not from the fact that W was Brownian, but from the regenerative structure of S. It is this that allows us to replace W by a more general, say a Lévy process, as long as we maintain the finite variance assumptions. The latter hold once we add a strictly negative drift to a zero-mean Lévy process W, reflect it, precisely as in (4), and integrate just as in (5). Whereas W may be discontinuous, S is continuous and the conclusion remains the same.

EXAMPLE 3 The third example is an application of the above in proving a limit theorem for a random digraph. We consider a random directed graph $G_n = (V_n, E_n)$ on the set of vertices $V_n := \{1, \ldots, n\}$ by letting the set of edges E_n contain the pair (i, j), $i < j$, with probability p, independently from pair to pair. This is a directed version of the (nowadays) so-called Erdős-Rényi graph.

A path starting in i and ending in j is a sequence of vertices $i_0 = i, i_1, \ldots, i_n = j$ such that $(i, i_1), \ldots (i_{n-1}, j)$ are edges. Amongst all paths in G_n there is one with maximum length; this length is denoted by L_n. Amongst all paths in G_n that end at a vertex $j \in V_n$ there is one with maximum length; this length is called *weight* of vertex j. We keep track of vertices with a specific weight and let $S_n(\ell)$ be the number of vertices with weights *at least* ℓ. (Here ℓ ranges between 0 and L_n.) So, for example, $S_n(0)$ is the number of vertices in V_n that are endpoints of no edge in E_n, and $S_n(L_n)$ is the number of paths of maximal length in G_n.

Theorem 3.
$$\frac{S_n([tL_n]) - tn}{\sqrt{n}}, \quad 0 \le t \le 1,$$
converges, as $n \to \infty$, weakly to a Brownian bridge.

The proof of this theorem can be found in [2, p. 453].

EXAMPLE 4 Here is an illustration, of the kind of phenomenon described around (3), in Stochastic Geometry. We consider a Poisson point process[1] N in $\mathbb{R}^d$ with intensity, say, 1; that is, N is a random discrete subset of $\mathbb{R}^d$ such that the cardinalities of $N \cap B_1, \ldots, N \cap B_n$ are independent random variables whenever $B_1, \ldots, B_n$ are disjoint Borel sets, for any $n \in \mathbb{N}$, and the expectation of the cardinality of $N \cap B$ equals the Lebesgue measure of B. For each x in $\mathbb{R}^d$ we let $\pi(x)$ be the point of N closest to x (there is a.s. a unique such point). For each point z of N, we let $\sigma(z)$ be the *Voronoi cell* [5, 6] associated to z:

$$\sigma(z) := \{x \in \mathbb{R}^d : \ ||x - z|| \le ||x - z'|| \text{ for all points } z' \text{ of } N\},$$

where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^d$. The *Voronoi tessellation* of $\mathbb{R}^d$ is the the tiling of $\mathbb{R}^d$ by the Voronoi cells. If z is not a point of N we define $\sigma(z)$ to be the Voronoi cell containing z (again this cell is a.s. unique). The distance of a closed set $A \subset \mathbb{R}^d$ from a point $x \in \mathbb{R}^d$ is
$$\mathrm{dist}(A, x) = \inf\{||x - y|| : \ y \in A\}.$$
Consider now the process
$$D(t, x) := \mathrm{dist}(\sigma(t\pi(x)), tx),$$
where $t \in [0, 1]$ and $x \in \mathbb{R}^d$. The claim is that

$$||x||^{-1/2} \ D(\cdot, x) \Rightarrow |W^0|, \quad \text{as } ||x|| \to \infty,$$

$|W^0|$ being the absolute value of a Brownian bridge.

[1] More general point processes can be allowed here.

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School of Mathematical and Computer Sciences
Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Edinburgh EH14 4AS, U.K.
foss@ma.hw.ac.uk, takis@ma.hw.ac.uk